1.a)

$$x(t) = 2\cos^2(2t) + \sin\left(\frac{t}{5}\right)$$

Let $x_1(t) = 2\cos^2(2t)$ and $x_2(t) = \sin\left(\frac{t}{5}\right)$

$= 1 + \cos\left(2\pi\left(\frac{2}{\pi}\right)t\right)$ $\qquad = \sin\left(2\pi\left(\frac{1}{10\pi}\right)t\right)$

By inspection, x_1 is $\frac{\pi}{2}$-periodic ($T_1 = \frac{\pi}{2}$) and $x_2(t)$ is 10π-periodic ($T_2 = 10\pi$)

These are the fundamental periods of x_1 and x_2; they also have periods that are integer multiples of the fundamental periods. We must find what the integer multipliers are to make them equal.

$$x(t+T_0) = x_1(t+T_0) + x_2(t+T_0) = x_1(t+nT_1) + x_2(t+mT_2)$$

$$nT_1 = mT_2 \rightarrow n\frac{\pi}{2} = m\,10\pi \rightarrow n = 20m$$

The smallest T_0 is clearly with $n=20, m=1$, so $\boxed{T_0 = 10\pi}$

Now write x as a sum of exponentials,

$$x(t) = 2\cos^2(2t) + \sin\left(\frac{t}{5}\right) = 1 + \cos(4t) + \sin\left(\frac{t}{5}\right)$$

$$= 1 + \cos\left(2\pi\frac{20}{T_0}t\right) + \sin\left(2\pi\frac{1}{T_0}t\right)$$

$$= 1 + \frac{1}{2}e^{2\pi j 20 t/T_0} + \frac{1}{2}e^{-2\pi j 20 t/T_0} + \frac{1}{2j}e^{2\pi j t/T_0} - \frac{1}{2j}e^{-2\pi j t/T_0}$$

$$\boxed{X[k] = \begin{cases} 1 & k=0 \\ \frac{1}{2} & k = \pm 20 \\ \pm\frac{1}{2j} & k = \pm 1 \\ 0 & \text{else} \end{cases}}$$

← the signs match, as in $X[-1] = -\frac{1}{2}$

1.b)

$$y(t) = j\cos(\sqrt{2}t) + \sin(2t)$$

Let $y_1(t) = j\cos(\sqrt{2}t)$ and $y_2(t) = \sin(2t)$

$$T_1 = \sqrt{2}\pi, \quad T_2 = \pi$$

$T_0 = nT_1 = mT_2 \Rightarrow n\sqrt{2}\pi = m\pi \rightarrow \frac{m}{n} = \sqrt{2} \rightarrow$ impossible, $\sqrt{2}$ is irrational (see appendix)

y is not periodic, thus it doesn't have a Fourier series.

1, appendix

$\sqrt{2} \neq \frac{m}{n}$ for all possible integers n, m

Suppose it is true: $n\sqrt{2} = m \rightarrow n^2 2 = m^2$

We can uniquely decompose n and m into prime factors.

$n = 2^{n_2} 3^{n_3} 5^{n_5} \ldots \qquad m = 2^{m_2} 3^{m_3} 5^{m_5} \ldots$

For example, $63 = 3^2 7$, so $n_2 = 0, n_3 = 2, n_5 = 0, n_7 = 1, n_i = 0, i > 7$

$n^2 2 = m^2$

$2 \cdot 2^{2n_2} 3^{2n_3} \ldots = 2^{2m_2} 3^{2m_3} \ldots$

$2^{2n_2+1} 3^{2n_3} \ldots = 2^{2m_2} 3^{2m_3}$

Since the prime decomposition is unique, $2n_2 + 1 = 2m_2$

In other words, an odd number equals an even number, a contradiction.

Thus $\sqrt{2} \neq \frac{m}{n}$.

2.a.)

$$x(t - T_0/2) = -x(t) \xrightarrow{\tilde{\mathcal{F}}_{T_0}} X[k]e^{-2\pi j k\, T_0/(2T_0)} = -X[k]$$

$$\rightarrow \quad e^{-\pi j k} X[k] + X[k] = 0$$

$$\rightarrow \quad 0 = \left((-1)^k + 1\right) X[k] = \begin{cases} 2X[k] & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

This says nothing about k odd. But it implies that $X[k] = 0$ for even k.

$$X[k] = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-2\pi j k t/T_0}\,dt = \frac{1}{T_0}\int_0^{T_0/2} x(t) e^{-2\pi j k t/T_0}\,dt + \frac{1}{T_0}\int_{T_0/2}^{T_0} x(t) e^{-2\pi j k t/T_0}\,dt$$

$$\int_{T_0/2}^{T_0} x(t) e^{-2\pi j k t/T_0}\,dt$$

$$= \int_{T_0}^{T_0/2} -x(t) e^{-2\pi j k t/T_0}\,dt = \int_{T_0}^{T_0/2} x\left(t - \frac{T_0}{2}\right) e^{-2\pi j k t/T_0}\,dt \qquad \begin{aligned} u &= t - \tfrac{T_0}{2} \\ du &= dt \end{aligned}$$

$$= \int_{T_0/2}^{0} x(u) e^{-2\pi j k (u + T_0/2)/T_0}\,du$$

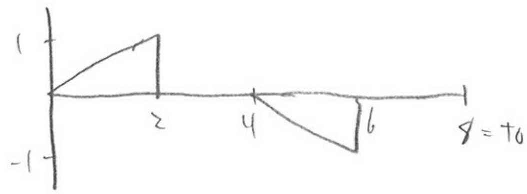
$$= -e^{-2\pi j k\, T_0/(2T_0)} \int_0^{T_0/2} x(u) e^{-2\pi j k u/T_0}\,du$$

$$= -(-1)^k \int_0^{T_0/2} x(u) e^{-2\pi j k u/T_0}\,du = \int_0^{T_0/2} x(u) e^{-2\pi j k t/T_0} \quad \text{for } k \text{ odd}$$

$$X[k] = \frac{1}{T_0}\int_0^{T_0/2} x(t) e^{-2\pi j k t/T_0}\,dt + \frac{1}{T_0}\int_0^{T_0/2} x(u) e^{-2\pi j k u/T_0}\,du$$

$$= \frac{2}{T_0}\int_0^{T_0/2} x(t) e^{-2\pi j k t/T_0}\,dt \quad (k \text{ odd})$$

2.b



$\boxed{X[k] = 0 \text{ if } k \text{ is even}}$

$$X[k] = \frac{2}{T_0}\int_0^{T_0/2} x(t) e^{-2\pi j k t/T_0}\, dt \quad \text{if } k \text{ is odd}$$

$$= \frac{2}{8}\int_0^2 \frac{t}{2} e^{-2\pi j k t/8}\, dt$$

$u = \frac{t}{2}$ $\quad dv = \frac{1}{4} e^{-\pi j k t/4} dt$

$du = \frac{1}{2} dt$ $\quad v = \frac{1}{4} \frac{e^{-\pi j k t/4}}{-\pi j k/4}$

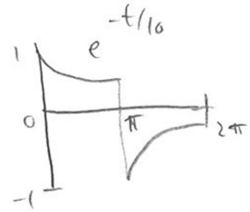
$$= \left.\frac{t}{2} \frac{e^{-\pi j k t/4}}{-\pi j k}\right]_{t=0}^{t=2} - \int_0^2 \frac{1}{2} \frac{e^{-\pi j k t/4}}{-\pi j k} dt$$

$$= \frac{e^{-\pi j k 2/4}}{-\pi j k} + -\frac{1}{2} \left.\frac{e^{-\pi j k t/4}}{(\pi j k)^2/4}\right]_{t=0}^{2}$$

$$= \frac{e^{-\pi j k/2}}{-\pi j k} - 2 \frac{e^{-\pi j k/2} - 1}{(\pi j k)^2}$$

$$= \boxed{j\frac{(-j)^k}{\pi k} + 2 \frac{(-j)^k - 1}{(\pi k)^2} \quad (k \text{ odd})}$$

2.6 cont



$X(k) = 0$ for k even

$$X[k] = \frac{2}{2\pi}\int_0^{\pi} e^{-t/10} e^{-2\pi jkt/2\pi} dt$$

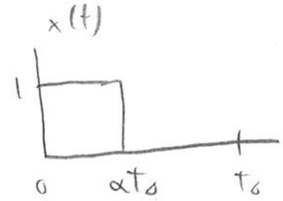
$$= \frac{1}{\pi}\int_0^{\pi} e^{-(jk + 1/10)t} dt$$

$$= \frac{1}{\pi} \left. \frac{e^{-(jk+1/10)t}}{-(jk + 1/10)} \right]_{t=0}^{\pi} = \frac{1}{2\pi} \frac{e^{-(jk+1/10)\pi} - 1}{-(jk + 1/10)}$$

$$= \frac{1}{\pi} \frac{1 - e^{-\pi/10}(-1)}{(jk + 1/10)} \quad (k \text{ is odd, so } e^{-jk\pi} = -1)$$

$$= \boxed{\frac{1}{\pi} \frac{1 + e^{-\pi/10}}{jk + 1/10} = X[k], \ k \text{ odd}}$$

3.



$0 < \alpha < 1$

I could do it with the Fourier series analysis integral, but I'll use Poisson Summation instead.

$$y(t) = \Pi\left(\frac{t - \frac{\alpha t_0}{2}}{\alpha t_0}\right) \qquad x(t) = \sum_{n=-\infty}^{\infty} y(t - nT_0)$$

$$\Pi(t) \xrightarrow{\tilde{\mathcal{F}}} \operatorname{sinc}(f)$$

$$\Pi\left(\frac{t}{\alpha t_0}\right) \xrightarrow{\tilde{\mathcal{F}}} \alpha t_0 \operatorname{sinc}(\alpha T_0 f)$$

$$\Pi\left(\frac{t - \frac{\alpha t_0}{2}}{\alpha t_0}\right) \xrightarrow{\tilde{\mathcal{F}}} \alpha t_0 \operatorname{sinc}(\alpha t_0 f)\, e^{-2\pi j f \left(\frac{\alpha t_0}{2}\right)}$$

$$X[k] = \frac{1}{t_0} Y\left(\frac{k}{t_0}\right) = \frac{1}{T_0} \alpha t_0 \operatorname{sinc}\left(\alpha T_0 \frac{k}{t_0}\right) e^{-2\pi j \frac{k}{t_0} \frac{\alpha T_0}{2}}$$

$$= \boxed{\alpha \operatorname{sinc}(\alpha k)\, e^{-\pi j \alpha k}}$$

4.a

Our filter is an LTI system, so it is completely described by its impulse response. The transfer function is the Fourier transform of the impulse response, and for all practical purposes, it is unique, and also completely determines the system. The output of the filter with an input of $e^{2\pi jft}$ is $H(f)e^{2\pi jft}$, where $H(f)$ is the transfer function. So based on the description,

$$y(t) = H(f)e^{2\pi jft} = e^{2\pi jft} \quad \text{if } |f| < 10$$

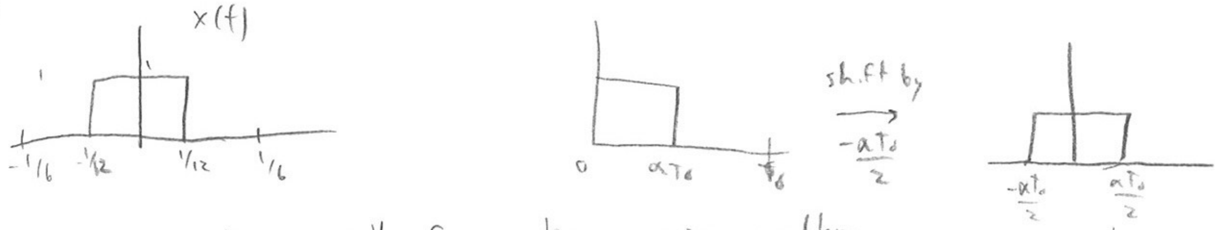
$$y(t) = H(f)e^{2\pi jft} = 0 \quad \text{if } |f| > 10$$

$$\text{so} \quad H(f) = \begin{cases} 1 & |f| < 10 \\ 0 & |f| > 10 \end{cases} = \boxed{\Pi\left(\frac{f}{20}\right) = H(f)}$$

$$\Pi(f) \xrightarrow{\tilde{\mathcal{F}}^{-1}} \operatorname{sinc}(t)$$

$$\Pi\left(\frac{f}{20}\right) \xrightarrow{\tilde{\mathcal{F}}^{-1}} \boxed{20\operatorname{sinc}(20t) = h(t)}$$

4.a)



we can use the result from the previous problem

$$X[k] = \alpha \,\text{sinc}(\alpha k)\, e^{-\pi j \alpha k}\, e^{-2\pi j k\left(-\frac{\alpha T_0}{2}\right)/T_0}$$

$$= \alpha \,\text{sinc}(\alpha k)\, e^{-\pi j \alpha k}\, e^{+\pi j k \alpha}$$

$$= \alpha \,\text{sinc}(\alpha k) = \frac{1}{2}\,\text{sinc}\left(\frac{k}{2}\right) \qquad \left(\alpha = \frac{1}{2}\right)$$

Our square wave can be written as a Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]\, e^{+2\pi j k t/T_0} = \sum_{k=-\infty}^{\infty} X[k]\, e^{2\pi j\left(\frac{k}{T_0}\right)t} = \sum_{k=-\infty}^{\infty} X[k]\, e^{2\pi j f_k t}$$

$$f_k = \frac{k}{T_0} = 3k$$

Our filter is LTI, so we can look at each sinusoid individually. For $|f_k| > 10$, the output is zero, for $|f_k| < 10$, the output is unchanged from the input.

$$y(t) = \sum_{k=-3}^{3} X[k]\, e^{+2\pi j f_k t} = X[0] + \sum_{k=1}^{3} X[k]\, 2\cos(2\pi f_k t) \qquad (X[k] \text{ is even})$$

$$= \frac{1}{2} + \text{sinc}\left(\frac{1}{2}\right)\cos(2\pi 3t) + \cancelto{0}{\text{sinc}(1)}\cos(2\pi 6t) + \text{sinc}\left(\frac{3}{2}\right)\cos(2\pi 9t)$$

$$= \boxed{\frac{1}{2} + \frac{2}{\pi}\cos(2\pi 3t) + -\frac{3}{2\pi}\cos(2\pi 9t) = y(t)}$$

In[251]:=

```
Sinc[t_] := Sin[π t] / (π t);
Sinc[0] := 1;
```

In[312]:=

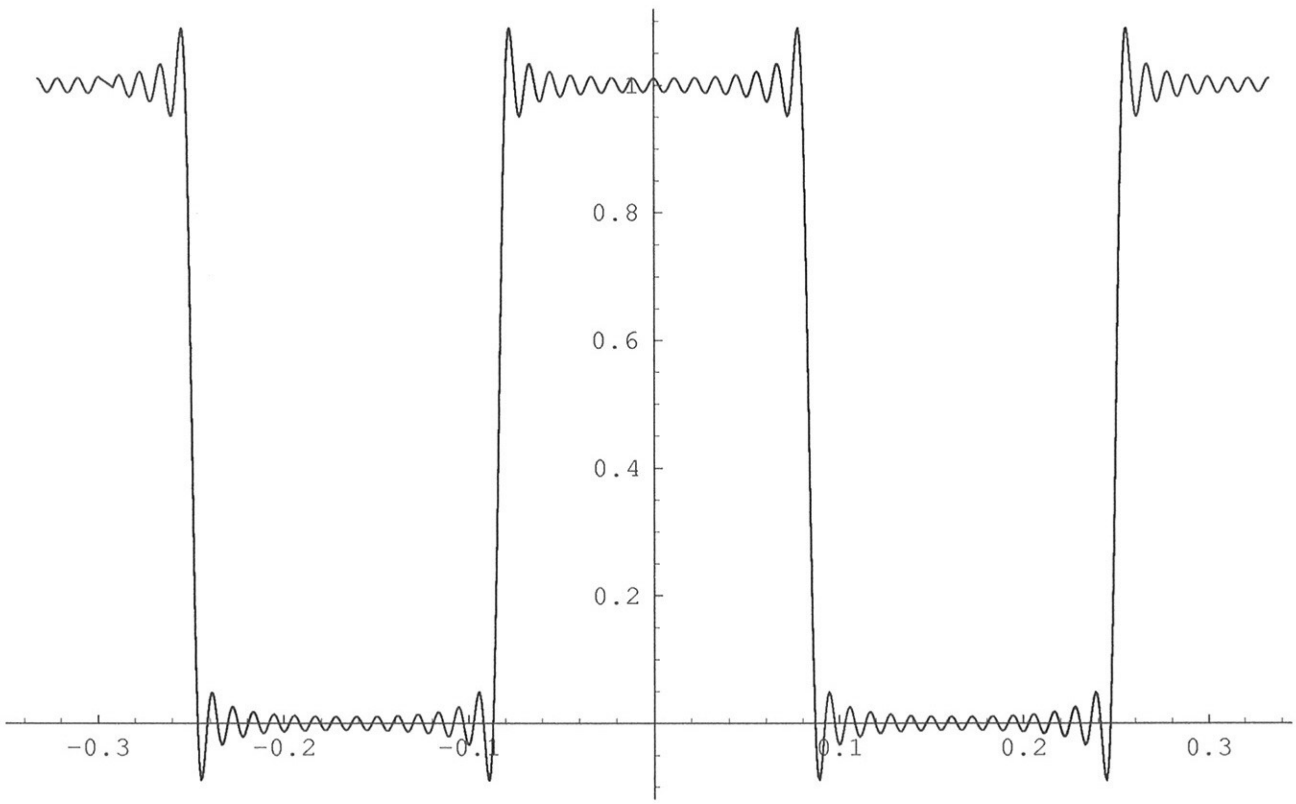
```
Clear[X];
α = 0.5;
X[k_] := α Sinc[α k];
T0 = 1 / 3;
KK = 30;
```

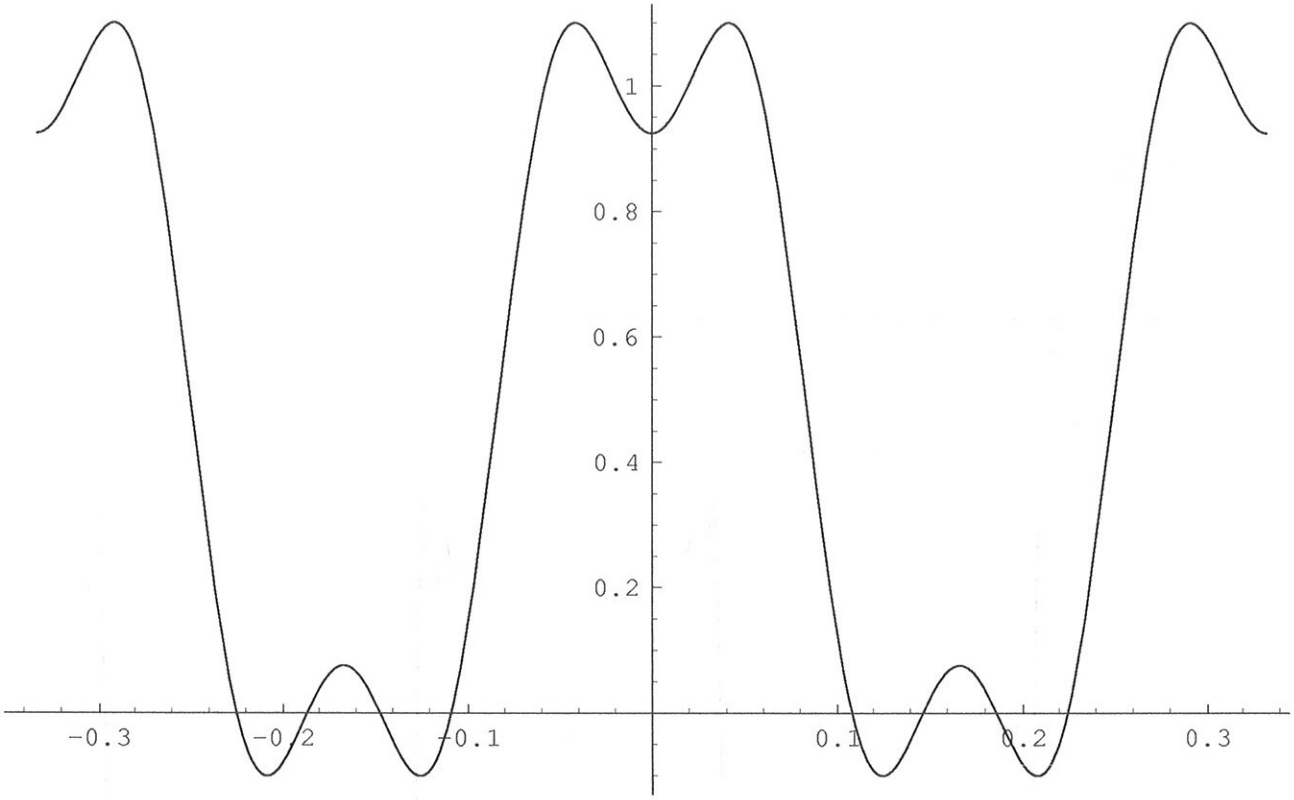
In[330]:=

```
x[t_] := Evaluate[Sum[X[k] Exp[2 π ⅈ k t / T0], {k, -KK, KK}]];
y[t_] := Evaluate[Sum[X[k] Exp[2 π ⅈ k t / T0], {k, -3, 3}]];
```

In[334]:=

```
Plot[x[t], {t, -T0, T0}, ImageSize → 600];
Plot[y[t], {t, -T0, T0}, ImageSize → 600];
```




1
0.8
0.6
0.4
0.2
-0.3
-0.2
-0.1
0.1
0.2
0.3